23/3/22

Application of Integral Answer sheet

1. Second fundamental Theorem of Integral Calculus:

If $f(x)$ be a continuous function defined on a closed interval $[a, b]$ and $F(x)$ is an anti-derivative of $f(x)$, then,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Property 1: $\int_a^b f(x)\,dx = \int_a^b f(u)\,du,\ a < b$

Property 2: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Property 3: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,\ a < c < b$

Property 4: $\int_a^b [\alpha f(x) + \beta g(x)]\,dx = \alpha\int_a^b f(x)\,dx + \beta\int_a^b g(x)\,dx$

where, α, β are constants

Property 5: If $x = g(u)$, then $\int_a^b f(x)\,dx = \int_c^d f(g(u))\,\frac{dg(u)}{du}\,du$

where $g(c) = a$ and $g(d) = b$

4) $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

Now, on putting $x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

x	0	1
θ	0	$\pi/4$

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\sec^2\theta\,d\theta$$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$I = \int_0^{\pi/4} \log\left(1+\tan\left(\pi/4-\theta\right)\right) d\theta$$

$$\boxed{\tan(x-y) = \frac{\tan x - \tan y}{1+\tan x \tan y}}$$

$$= \int_0^{\pi/4} \log\left(\frac{1+\tan\theta + 1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$I = \log \int_0^{\pi/4} d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

$$2I = \frac{\pi}{4}\log 2 \Rightarrow I = \frac{\pi}{8}\log 2 \;//$$

5. $$\int_0^{\pi/2} \frac{dx}{5+4\sin^2 x}$$

x	0	$\pi/2$
u	0	∞

$$I = \int_0^{\pi/2} \frac{dx}{5\cos^2 x + 5\sin^2 x + 4\sin^2 x}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\,dx}{5+9\tan^2 x}$$

$$\boxed{\begin{aligned} u &= \tan x \\ du &= \sec^2 x\,dx \end{aligned}}$$

$$I = \frac{1}{9}\int_0^{\infty} \frac{du}{u^2+5/9}$$

$$= \left[\frac{1}{3\sqrt{5}} \tan^{-1}\frac{3u}{\sqrt{5}}\right]_0^{\infty}$$

$$= \frac{1}{3\sqrt{5}}\left[\tan^{-1}\infty - \tan^{-1} 0\right]$$

$$= \frac{1}{3\sqrt{5}}\,\frac{\pi}{2} = \frac{\pi}{6\sqrt{5}} \;//$$

6. $\int_0^{\pi/2} \sin^3\theta \cos^5\theta \, d\theta$

$$I = \int_0^{\pi/2} (1-\cos^2\theta)\cos^5\theta \sin\theta \, d\theta$$

$$I = \int_0^{\pi/2} (\cos^5\theta - \cos^7\theta)\sin\theta \, d\theta$$

x	0	$\pi/2$
t	1	0

$$\boxed{t = \cos\theta, \quad dt = -\sin\theta \, d\theta}$$

$$I = \int_0^1 (t^5 - t^7)(-dt)$$

$$= \int_0^1 (t^5 - t^7)(dt)$$

$$= \left[\frac{t^6}{6} - \frac{t^8}{8}\right]_0^1$$

$$= 1/6 - 1/8 = \frac{8-6}{48} = 1/24 \,//$$

2. $\int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x}) \, dx$

$$I = \int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x}) \, dx.$$

$$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} \, dx = \sqrt{2} \int_0^{\pi/2} \frac{(\sin x + \cos x) \, dx}{\sqrt{1 - (\sin x - \cos x)^2}}$$

$$\boxed{\text{Put } u = \sin x - \cos x, \text{ then } du = (\cos x + \sin x) \, dx}$$

$x = 0$, $u = -1$, when $x = \pi/2$, $u = 1$

$$I = \sqrt{2} \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} = \sqrt{2}\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= \pi\sqrt{2} \,//.$$

3. Show that

$$\int_0^{\pi/2} \frac{dx}{4+5\sin x} = \frac{1}{3}\log_e 2$$

Put $u = \tan\frac{x}{2}$

$$\sin x = \frac{2\tan x}{1+\tan^2 x/2} = \frac{2u}{1+u^2},$$

$$du = \frac{1}{2}\sec^2 x/2 \, dx \Rightarrow dx = \frac{2du}{1+u^2}$$

$\boxed{\text{When } x=0,\ u=\tan 0 = 0, \text{ when } x=\pi/2,\ u=\tan\pi/4=1}$

$$I = \int_0^{\pi/2} \frac{dx}{4+5\sin x} = \int_0^1 \frac{\frac{2du}{1+u^2}}{4+5\left(\frac{2u}{1+u^2}\right)}$$

$$= \frac{1}{2}\int_0^1 \frac{du}{u^2+\frac{5}{2}u+1} = \frac{1}{3}\left[\log\left[\frac{u+1/2}{u+2}\right]\right]_0^1$$

$$= \frac{1}{3}\log 2$$

7. $$I = \int_0^\infty e^{-\alpha x^2} x^3\,dx$$

x	0	∞
t	0	∞

$\boxed{\text{Put } t = x^2,\ dt = 2x\,dx}$

$$\frac{dt}{2} = x\,dx$$

$$= \int_0^\infty e^{-\alpha t}\, t\, \frac{dt}{2} = \frac{1}{2} \cancel{\times} \frac{1}{\alpha^2}$$

Given

$$\int_0^\infty e^{-\alpha x^2} x^3 \, dx = 32$$

$$\frac{1}{2} \times \frac{1}{\alpha^2} = 32$$

$$\frac{1}{\alpha^2} = 64 \qquad \alpha = \pm \tfrac{1}{8} = \tfrac{1}{8}$$ //

8. Given:

$$x + y = 3$$

$$y^2 - 4x$$

To find: Area of region in 1^{st} quadrant

Solu: $x + y = 3 \Rightarrow y = 3 - x$

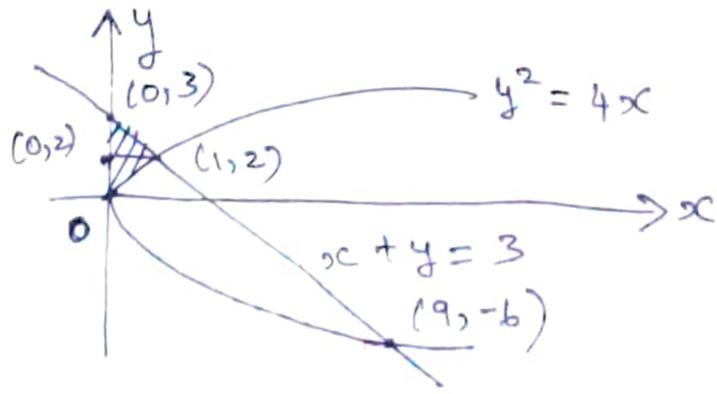
$$y^2 - 4x \Rightarrow y^2 = 4x \Rightarrow (3-x)^2 = 4x$$

$$\Rightarrow x^2 - 10x + 9 = 0$$

$$x = 1, \; x = 9$$

$x = 1$ in $x + y = 3$ $y = 2$	, $x = 9$ in $x + y = 3$ $y = -6$

$(1, 2)$ and $(9, -6)$ are the point of intersections



$$\boxed{x = \begin{cases} y^2/4, & 0 \le y \le 2 \\ 3-y, & 2 \le y \le 3 \end{cases}}$$

$$A = \int_0^2 x\,dy + \int_2^3 x\,dy = \int_0^2 \frac{y^2}{4}\,dy + \int_2^3 (3-y)\,dy$$

$$= \left(\frac{y^3}{12}\right)_0^2 + \left(3y - \frac{y^2}{2}\right)_2^3 = \left(\frac{8}{12} - 0\right) + \left(9 - \frac{9}{2}\right) - \left(6 - \frac{4}{2}\right)$$

$$= \frac{7}{6}$$

9. Given ÷

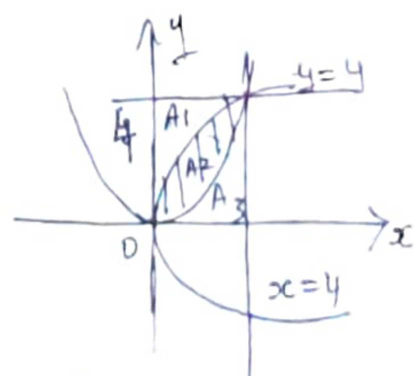
$y^2 = 4x$

$x^2 = 4y$ are two curves which splits the square field ($x=0$, $x=4$, $y=0$, $y=4$) into three equal areas for wife, son and daughter

To find ÷ Area divided among them

Area A_1 (Son) = Area A_2 (wife) = Area A_3 (daughter)

Solu÷



$y^2 = 4x$, $x^2 = 4y$

$$\left(\frac{y^2}{4}\right)^2 = 4y$$

$y^4 = 64y \Rightarrow y(y^3 - 64) = 0$

$y = 0$, $y = 4$

x	0	4
y	0	4

Point of intersection are O (0,0) and (4,4)

Area of region bounded

$$A_2 = \int_0^4 \left(\sqrt{4x} - \frac{x^2}{4}\right)dx$$

$\int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$

$= \left[\frac{4}{3}(4)^{3/2} - \frac{4^3}{12} - 0\right] = \frac{16}{3}$ sq. units

$A_3 = \int_0^4 y\,dx = \int_0^4 \left(\frac{x^2}{4}\right) dx$

$= \left[\frac{x^3}{12}\right]_0^4 = \frac{16}{3}$ sq. units

$A_1 = \int_0^4 x\,dy = \int_0^4 \left(\frac{y^2}{4}\right) dy$

$= \left[\frac{y^3}{12}\right]_0^4 = \frac{16}{3}$ sq. units

$A_1 = A_2 = A_3 = \frac{16}{3}$ sq. units

10. Given :

$x^2 + y^2 = 16$

$y^2 = 6x$

To find : find the area of region.

$x^2 + y^2 = 16$

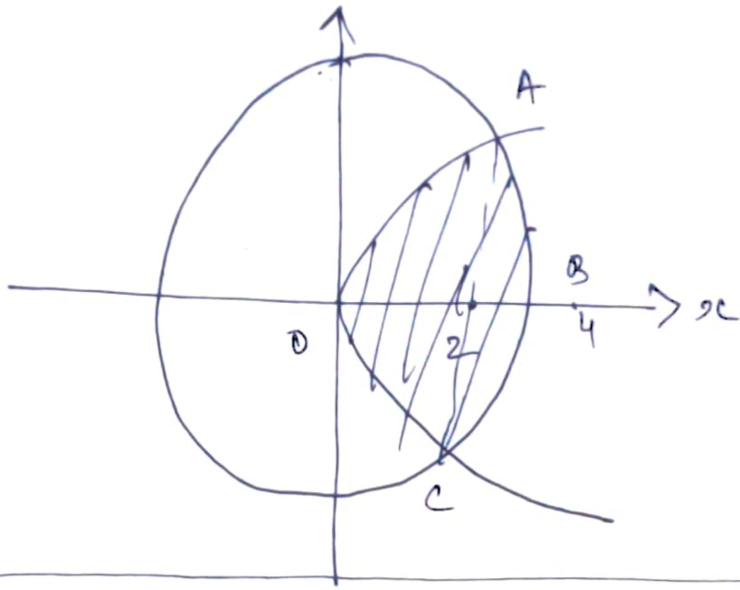
$y = 6x$

$x^2 + 6x - 16 = 0$

$(x+8)(x-2) = 0$ $\quad$ $x = -8$ impossible

$x = -8, 2$ $\quad$ $x = 2,\ y = 2\sqrt{3}$

Radius of circle $x^2 + y^2 = 16$ is 4



Area OABC = 2 (Area of OAB)

$= 2 [y^2 = 6x [0, 2], + (x^2 + y = 16)$

$= 2 \left[\int_0^2 \sqrt{6x}\, dx + \int_2^4 \sqrt{16 - x^2}\, dx \right]$

$= 2 \left[\frac{4}{3}\sqrt{12} + \frac{8\pi}{3} - \sqrt{12} \right]$

$= \frac{4}{3} [4\pi + \sqrt{3}]$ sq. units